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ON THE EXPONENTIAL DECAY OF STRESSES IN CIRCULAR ELASTIC CYLINDERS SUBJECT TO AXISYMMETRIC SELF-EQUILIBRATED END LOADS†

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Abstract—Methods involving energy-decay inequalities are applied to the axisymmetric end problem for a circular elastic cylinder. Explicit lower bounds in terms of Poisson's ratio are obtained for the rate of exponential decay of stresses, and these are compared with results of other authors.

1. INTRODUCTION

SEVERAL recent papers have examined questions connected with Saint-Venant's principle for elastic solids in an effort to establish the exponential decay of stresses away from a portion of the boundary which is subject to self-equilibrated surface tractions. Thus in [1], Toupin proved that a self-equilibrated load on one end of an elastic, anisotropic cylinder of arbitrary cross-section produces a state of deformation in which the strain energy stored beyond a distance z from the loaded end decays exponentially with z. He further showed that the stresses at interior points in such a cylinder also obey such an exponential law. The restriction of this latter result to strictly interior points was removed by Roseman [2], who obtained pointwise estimates valid up to the lateral surface of the cylinder.

In the linear theory of plane strain for isotropic elastic solids, Knowles [3] established the exponential decay of strain energy away from a portion of the boundary carrying a self-equilibrated load for a general class of domains. Stress estimates were given for interior points.

The decay of strain energy and stresses was examined by Knowles and Sternberg [4] for the problem of axisymmetric *torsion* of isotropic elastic solids of revolution.

The effect of self-equilibrated loads in nonlinear elasticity has been considered recently by Roseman [5].

In the investigation [3] pertaining to plane strain, a single explicit decay constant, valid for all domains of the type considered, was obtained. For axisymmetric torsion, the results of [4] provide the rate of decay in terms of the root of an explicit transcendental equation. In the more complicated case of the arbitrary cylinder as considered by Toupin in [1], the decay constant is characterized in such a way that its explicit estimation appears to be prohibitively difficult, even for the case of an isotropic circular cylinder.

While the arguments presented in [1], [3] and [4] all rely on energy-decay inequalities, they differ in many important details. The present paper has as its main purpose the

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determination of an energy-decay inequality for an isotropic circular cylinder of finite length in the presence of axisymmetry by an argument closely parallel to that used for the plane case in [3]. For such a cylinder, subject on one end to a torsionless axisymmetric self-equilibrated load, we prove that the strain energy decays exponentially away from the loaded end, and we determine a decay constant explicitly. Numerical values of this constant, which provides a lower bound for the actual rate of decay, are given for various values of Poisson's ratio and compared with decay constants obtained for this problem by other authors using approximate methods which could not be expected to generalize to other geometries [6, 7].

In the following section we state the boundary value problem to be considered in terms of stresses and displacements referred to cylindrical coordinates. In order to adapt the procedure used in [3] to the present case, it is helpful to reformulate the problem in terms of a pair of stress functions. This reformulation is described in Sections 3 and 4. Properties of the strain energy and of a related function which are required in our analysis are derived in Sections 5 and 6. The derivation of the exponential decay inequality for the strain energy is given in Section 7. Pointwise estimates for the stresses are described briefly in Section 8, and in Section 9 we discuss the decay constant furnished by our procedure and compare it with results of other authors.

2. THE BOUNDARY VALUE PROBLEM

We consider a circular cylinder of radius a and length l, and we use the natural cylindrical coordinates r, θ, z. For torsionless axisymmetric deformations of such a cylinder, the non-vanishing components of the displacement vector $\mathbf{u}$, the strain tensor $\mathbf{e}$ and the stress tensor $\boldsymbol{\tau}$ are respectively denoted by u_r, u_z; $e_{rr}, e_{\theta\theta}, e_{zz}, e_{rz}$; and $\tau_{rr}, \tau_{\theta\theta}, \tau_{zz}, \tau_{rz}$. The interior of the cylinder is designated by $\mathscr{R}$; a fixed meridional cross-section for which $0 < r < a$, $0 < z < l$ is denoted by $\mathscr{M}$.

Under the assumed condition of axisymmetry and in the absence of body forces, the field equations of linear elasticity for an isotropic homogeneous material may be written as follows.

Equilibrium equations:

$$r\frac{\partial \tau_{rr}}{\partial r}+r\frac{\partial \tau_{rz}}{\partial r}+\tau_{rr}-\tau_{\theta\theta} = 0, \tag{2.1}$$

$$r\frac{\partial \tau_{rz}}{\partial r}+r\frac{\partial \tau_{zz}}{\partial z}+\tau_{rz} = 0. \tag{2.2}$$

Stress–strain relations:

$$\tau_{rr} = 2\mu\left[e_{rr}+\frac{\sigma}{1-2\sigma}\left(e_{rr}+e_{\theta\theta}+e_{zz}\right)\right], \tag{2.3}$$

$$\tau_{\theta\theta} = 2\mu\left[e_{\theta\theta}+\frac{\sigma}{1-2\sigma}\left(e_{rr}+e_{\theta\theta}+e_{zz}\right)\right], \tag{2.4}$$

$$\tau_{zz} = 2\mu\left[e_{zz}+\frac{\sigma}{1-2\sigma}\left(e_{rr}+e_{\theta\theta}+e_{zz}\right)\right], \tag{2.5}$$

$$\tau_{zr} = \tau_{rz} = \mu e_{rz}. \tag{2.6}$$

Strain–displacement relations:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \tag{2.7}$$

$$re_{\theta\theta} = u_r, \tag{2.8}$$

$$e_{zz} = \frac{\partial u_z}{\partial z}, \tag{2.9}$$

$$e_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}. \tag{2.10}$$

We assume that the shear modulus μ and Poisson's ratio σ satisfy the inequalities $\mu > 0$, $-1 < \sigma < \frac{1}{2}$.

The boundary conditions to accompany the field equations listed above are as follows:

$$r = a: \quad \tau_{rz}(a, z) = \tau_{rr}(a, z) = 0, \quad 0 \leq z \leq l; \tag{2.11}$$

$$z = 0: \quad \tau_{zz}(r, 0) = f(r), \quad \tau_{zr}(r, 0) = g(r), \quad 0 \leq r \leq a; \tag{2.12}$$

$$z = l: \quad \tau_{zz}(r, l) = \tau_{zr}(r, l) = 0, \quad 0 \leq r \leq a. \tag{2.13}$$

The conditions imposed at $z = 0$ correspond to prescribed axisymmetric normal and shear tractions. A necessary condition for the existence of an equilibrium state in the cylinder is the vanishing of the total axial force:

$$2\pi \int_0^a rf(r)\,\mathrm{d}r = 0. \tag{2.14}$$

The given functions f and g are assumed to be continuously differentiable on $[0, a]$.

We seek a displacement field $\mathbf{u}$ which is twice continuously differentiable on the closed meridional cross section $\bar{\mathscr{M}}$, stress and strain fields $\boldsymbol{\tau}$, $\mathbf{e}$ which are continuously differentiable on $\bar{\mathscr{M}}$, and the governing equations (2.1) to (2.10) are required to hold on $\bar{\mathscr{M}}$.

There are three necessary conditions for the existence of displacements, stresses and strains satisfying (2.1)–(2.13) which will be of importance in the subsequent development. First, it follows from (2.2) and the assumed smoothness of the stress field that

$$\tau_{rz}(r, z) = O(r) \quad \text{as } r \to 0, \quad 0 \leq z \leq l \tag{2.15}$$

uniformly in z. Similarly, we conclude from (2.8) that the radial displacement satisfies

$$u_r(r, z) = O(r) \quad \text{as } r \to 0, \quad 0 \leq z \leq l, \tag{2.16}$$

uniformly in z. Finally, it follows from the second of (2.12), the first of (2.11), and the symmetry and continuity of $\boldsymbol{\tau}$ on $\bar{\mathscr{M}}$, that

$$g(a) = 0. \tag{2.17}$$

From (2.15) and the second of (2.12) it further follows that

$$g(r) = O(r) \quad \text{as } r \to 0. \tag{2.18}$$

It is assumed that the given functions f and g satisfy (2.14), (2.17) and (2.18).

Any two solutions of the boundary-value problem posed above differ at most by an axisymmetric rigid body displacement.

3. STRESS FUNCTIONS

In the course of deriving expressions for stresses and displacements in terms of a single biharmonic function in the presence of axisymmetry, Love [8] shows that u_r and u_z can be represented in the form

$$u_r = -\frac{1}{2\mu}(\Omega_r + \varphi_r), \qquad u_z = \frac{1}{2\mu}(\Omega_z - \varphi_z), \tag{3.1}$$

where the subscripts r and z on Ω and φ indicate partial derivaties,† and where Ω and φ satisfy the differential equations

$$\nabla^2\Omega \equiv \Omega_{rr} + \frac{1}{r}\Omega_r + \Omega_{zz} = 0, \tag{3.2}$$

$$(1-\sigma)\nabla^2\varphi = \Omega_{zz}. \tag{3.3}$$

For our purposes it is convenient to introduce the function χ which is conjugate to Ω in the sense that

$$r\Omega_z = \chi_r, \qquad r\Omega_r = -\chi_z, \tag{3.4}$$

so that (3.1) may be written in the form

$$ru_r = \frac{1}{2\mu}(\chi_z - r\varphi_r), \qquad ru_z = \frac{1}{2\mu}(\chi_r - r\varphi_z). \tag{3.5}$$

The stresses following from (3.5) and (2.3) through (2.10) are easily computed. They are given by

$$r^2\tau_{rr} = r^2\varphi_{zz} + r\varphi_r - \chi_z, \tag{3.6}$$

$$r^2\tau_{\theta\theta} = \sigma r^2\varphi_{zz} + \sigma r^2\varphi_{rr} - (1-\sigma)r\varphi_r + \chi_z, \tag{3.7}$$

$$r\tau_{zz} = (r\varphi_r)_r, \tag{3.8}$$

$$\tau_{rz} = -\varphi_{rz}. \tag{3.9}$$

The representations (3.8) and (3.9) are closely analogous to corresponding formulas in terms of Airy's stress function in plane elasticity, and it is mainly this analogy which makes it possible to adapt the procedure used in [3].

A virtual retracing of the steps of Love's argument establishes the following statement. A solution **u**, **e**, **τ** of the field equations (2.1)–(2.10), with the smoothness properties stated in the preceding section, exists if and only if there exist functions φ and χ, three times

† In the sequel, partial differentiation of all functions *except* stresses, strains and displacements will be indicated by subscripts.

continuously differentiable on $\overline{\mathscr{M}}$, satisfying the differential equations

$$(1-\sigma)r\nabla^2\varphi = \chi_{rz}, \tag{3.10}$$

$$r\chi_{rr}-\chi_r+r\chi_{zz} = 0 \tag{3.11}$$

on the closed meridional section† $\overline{\mathscr{M}}$.

If φ and χ are chosen in the special form

$$\varphi = \alpha+\beta\log r+\gamma z, \qquad \chi = \beta z+\delta+\eta r^2, \tag{3.12}$$

where α, β, γ, δ, η are constants, it may be verified through (3.6)–(3.9) that the corresponding stresses vanish identically, and that (3.10) and (3.11) are satisfied. It follows that the special choice (3.12) of φ and χ corresponds to a rigid body displacement. Conversely, it is easily shown that (3.12) represents the most general axisymmetric rigid body displacement.

4. THE BOUNDARY VALUE PROBLEM FOR φ, χ

If the stresses given in terms of φ and χ by (3.6) through (3.9) are to satisfy the boundary conditions (2.11) through (2.13), then φ and χ must satisfy the following boundary conditions.

$$r = a: \qquad \varphi_{rz} = 0, \qquad 0 \le z \le l, \tag{4.1}$$

$$r = a: \qquad a^2\varphi_{zz}+a\varphi_r-\chi_z = 0, \qquad 0 \le z \le l. \tag{4.2}$$

$$z = l: \qquad \varphi_{rz} = 0, \qquad 0 \le r \le a, \tag{4.3}$$

$$z = l: \qquad (r\varphi_r)_r = 0, \qquad 0 \le r \le a. \tag{4.4}$$

$$z = 0: \qquad \varphi_{zr} = -g, \qquad 0 \le r \le a, \tag{4.5}$$

$$z = 0: \qquad (r\varphi_r)_r = rf, \qquad 0 \le r \le a. \tag{4.6}$$

The order conditions (2.15) and (2.16), together with (3.9) and (3.5) require

$$\varphi_{rz} = O(r) \quad \text{as } r \to 0, \tag{4.7a}$$

$$\chi_z-r\varphi_r = O(r^2) \quad \text{as } r \to 0, \tag{4.7b}$$

uniformly in z, $0 \le z \le l$.

It follows from (4.1) that

$$\varphi_r(a, z) = c_1, \qquad 0 \le z \le l \tag{4.8}$$

and hence from (4.2) and (4.8) that

$$a^2\varphi_z(a, z)-\chi(a, z) = -ac_1z+c_2, \qquad 0 \le z \le l \tag{4.9}$$

where c_1 and c_2 are constants. From (4.3) and (4.4)

$$\varphi_z(r, l) = c_3, \qquad 0 \le r \le a, \tag{4.10}$$

$$r\varphi_r(r, l) = c_4, \qquad 0 \le r \le a, \tag{4.11}$$

† It has been pointed out to us by E. Sternberg that the representation (3.5) in terms of potentials χ and φ can be derived from Boussinesq's solution for axisymmetric problems. (See [9] for a discussion of the latter.) It is possible to apply the representation (3.5) for axisymmetric bodies of more general shape than the circular cylinder being considered here.

where c_3 and c_4 are constants. Equations (4.5), (4.6) imply

$$\varphi_z(r,0) = -\int_0^r g(\rho)\,\mathrm{d}\rho + c_5, \qquad 0 \le r \le a, \tag{4.12}$$

$$r\varphi_r(r,0) = \int_0^r \rho f(\rho)\,\mathrm{d}\rho + c_6, \qquad 0 \le r \le a; \tag{4.13}$$

c_5 and c_6 are constants. Integrating the uniform order condition (4.7a) with respect to z from z to l, and using (4.11), we find that

$$\varphi_r(r,z) = \frac{c_4}{r} + O(r) \quad \text{as } r \to 0, \tag{4.14}$$

uniformly for $0 \le z \le l$.

For continuity of φ_r at $r = a$, $z = l$, it is necessary that $c_4 = ac_1$. In view of (2.14), (4.8) and (4.13), it is further necessary that $c_6 = ac_1$ to assure the continuity of φ_r at $r = a$, $z = 0$.

In order to show that the constants of integration introduced above are inessential, we introduce $\tilde{\varphi}$ and $\tilde{\chi}$ through

$$\left.\begin{aligned} \varphi &= \tilde{\varphi} + ac_1 \log r + c_3 z, \\ \chi &= \tilde{\chi} + ac_1 z + c_3 r^2 - c_2. \end{aligned}\right\} \tag{4.15}$$

In view of (3.12), $\tilde{\varphi}$ and $\tilde{\chi}$ differ from φ and χ by terms which correspond to a rigid body displacement. Equations (4.8)–(4.13) can now be converted to conditions on $\tilde{\varphi}$, $\tilde{\chi}$. Recalling $c_4 = ac_1$, $c_6 = ac_1$, we have

$$r = a: \qquad \tilde{\varphi}_r = 0, \quad a^2\tilde{\varphi}_z - \tilde{\chi} = 0, \tag{4.16}$$

$$z = l: \qquad \tilde{\varphi}_z = 0, \quad \tilde{\varphi}_r = 0, \tag{4.17}$$

$$z = 0: \qquad \tilde{\varphi}_z = G(r), \quad \tilde{\varphi}_r = F(r), \tag{4.18}$$

where

$$G(r) = c_5 - c_3 - \int_0^r g(\rho)\,\mathrm{d}\rho, \tag{4.19}$$

$$F(r) = \frac{1}{r}\int_0^r \rho f(\rho)\,\mathrm{d}\rho. \tag{4.20}$$

At the axis, we have from (4.14), (4.15)

$$\tilde{\varphi}_r = O(r) \quad \text{as } r \to 0. \tag{4.21}$$

In terms of $\tilde{\varphi}$, $\tilde{\chi}$, (4.7b) becomes

$$\tilde{\chi}_z - r\tilde{\varphi}_r = O(r^2) \quad \text{as } r \to 0,$$

and thus, by (4.21)

$$\tilde{\chi}_z = O(r^2) \quad \text{as } r \to 0. \tag{4.22}$$

To obtain further estimates at the axis, we return to (4.7a) and express it in terms of $\tilde{\varphi}$:

$$\tilde{\varphi}_{zr} = O(r) \quad \text{as } r \to 0.$$

Integration with respect to r leads to the conclusion that

$$\tilde{\varphi}_z = O(1) \quad \text{as } r \to 0. \tag{4.23}$$

Next, an order estimate for $\tilde{\chi}_r$ at the axis follows from (4.23), (3.5) and the continuity of u_z on $\overline{\mathscr{M}}$. Thus

$$\tilde{\chi}_r = O(r) \quad \text{as } r \to 0. \tag{4.24}$$

Finally it is easy to prove that

$$\tilde{\varphi} = O(1) \quad \text{as } r \to 0. \tag{4.25}$$

The order estimates (4.21)–(4.25) are all uniform in z for $0 \leq z \leq l$.

Since the second of (4.17) implies that $\tilde{\varphi}(r, l) =$ constant for $0 \leq r \leq a$, and since an arbitrary constant can be added to $\tilde{\varphi}$ without affecting the differential equations or boundary conditions, we may assume that

$$\tilde{\varphi}(r, l) = 0, \qquad 0 \leq r \leq a. \tag{4.26}$$

We now summarize the boundary value problem in final form, dropping the tildes for convenience. We seek functions φ, χ, three times continuously differentiable on $\overline{\mathscr{M}}$, satisfying the following differential equations and boundary conditions.

$$r(1-\sigma)\nabla^2\varphi \equiv (1-\sigma)[(r\varphi_r)_r + r\varphi_{zz}] = \chi_{rz}, \quad \text{on } \overline{\mathscr{M}}, \tag{4.27}$$

$$r\chi_{rr} - \chi_r + r\chi_{zz} = 0, \quad \text{on } \overline{\mathscr{M}}, \tag{4.28}$$

$$r = a: \qquad \varphi_r = 0, \qquad 0 \leq z \leq l, \tag{4.29}$$

$$r = a: \qquad a^2\varphi_z - \chi = 0, \qquad 0 \leq z \leq l, \tag{4.30}$$

$$z = l: \qquad \varphi = 0, \qquad 0 \leq r \leq a, \tag{4.31}$$

$$z = l: \qquad \varphi_z = 0, \qquad 0 \leq r \leq a, \tag{4.32}$$

$$z = 0: \qquad \varphi_z = G, \qquad \varphi_r = F, \qquad 0 \leq r \leq a. \tag{4.33}$$

$$r \to 0: \quad \varphi_r = O(r), \quad \varphi_z = O(1), \quad \chi_r = O(r), \quad \chi_z = O(r^2), \quad \varphi = O(1), \tag{4.34}$$

uniformly in z for $0 \leq z \leq l$. The functions F and G are defined in (4.19), (4.20).

The order conditions (4.34) at $r = 0$ are sufficient to assure that the solution χ of (4.28) is infinitely differentiable in r and z *on the z-axis*. (See [10].) It may therefore be assumed that χ is infinitely differentiable for $0 \leq r < a$, $0 < z < l$. This differentiability in turn implies that the solution φ of (4.27) is infinitely differentiable for $0 \leq r < a$, $0 < z < l$. (See p. 345 of [11].)

Assuming the existence of a solution to the boundary value problem posed above, we now derive a "conservation property" of the solution. Integrating (4.27) with respect to r from $r = 0$ to $r = a$ for fixed z and using (4.29) and (4.34), we find that

$$(1-\sigma)\int_0^a r\varphi_{zz}(r, z)\,\mathrm{d}r = \chi_z(a, z).$$

Using (4.30) we may write this in the form

$$\frac{d^2}{dz^2}\left[(1-\sigma)\int_0^a r\varphi(r,z)\,dr - a^2\varphi(a,z)\right] = 0. \tag{4.35}$$

Integrating with respect to z and using (4.32), (4.31) we find

$$(1-\sigma)\int_0^a r\varphi(r,z)\,dr - a^2\varphi(a,z) = 0, \qquad 0 \le z \le l. \tag{4.36†}$$

5. STRAIN ENERGY AND ITS REPRESENTATION

Under the prevailing conditions of axisymmetry, the strain energy $U(\zeta)$ contained in that portion of the cylinder for which $\zeta \le z \le l$ is given by

$$U(\zeta) = 2\pi \int_\zeta^l \int_0^a W(\boldsymbol{\tau}) r\,dr\,dz, \qquad 0 \le \zeta \le l, \tag{5.1}$$

where

$$W(\boldsymbol{\tau}) = \frac{1}{4\mu}\left[\tau_{rr}^2 + \tau_{\theta\theta}^2 + \tau_{zz}^2 + 2\tau_{rz}^2 - \frac{\sigma}{1+\sigma}(\tau_{rr}+\tau_{\theta\theta}+\tau_{zz})^2\right]. \tag{5.2}$$

An alternate formula for $U(\zeta)$ is supplied by the work-energy relation which states that $2U(\zeta)$ is equal to the work done on the subcylinder for which $\zeta \le z \le l$ by the surface tractions acting on the boundary. Thus

$$2U(\zeta) = -2\pi \int_0^a [\tau_{zr}(r,\zeta)u_r(r,\zeta) + \tau_{zz}(r,\zeta)u_z(r,\zeta)]r\,dr. \tag{5.3}$$

To obtain a representation for $U(\zeta)$ in terms of the stress functions φ and χ, we proceed from (5.3) as follows. From (3.5), (3.8) and (3.9),

$$U(z) = -\frac{\pi}{2\mu}\int_0^a \left[\varphi_{rz}(r\varphi_r - \chi_z) + (r\varphi_r)_r\left(\frac{1}{r}\chi_r - \varphi_z\right)\right] dr, \tag{5.4}$$

where the derivatives of φ and χ in the integrand are evaluated at r, z. From (5.4) it follows that

$$\begin{aligned} -\frac{2\mu}{\pi}U'(z) = \int_0^a \Bigg[& \varphi_{rzz}(r\varphi_r - \chi_z) + \varphi_{rz}(r\varphi_{rz} - \chi_{zz}) + (r\varphi_{rz})_r\left(\frac{1}{r}\chi_r - \varphi_z\right) \\ & + (r\varphi_r)_r\left(\frac{1}{r}\chi_{rz} - \varphi_{zz}\right)\Bigg] dr, \end{aligned} \tag{5.5}$$

† By differentiating (4.36) and setting $z = 0$, it is possible to show that the constant $c_5 - c_3$ appearing in (4.19) is given by

$$c_5 - c_3 = \int_0^a \left(1 + \frac{1-\sigma}{1+\sigma}\frac{r^2}{a^2}\right) g(r)\,dr.$$

where the prime indicates differentiation with respect to z. Suitable integrations by parts then yield

$$
\begin{aligned}
-\frac{2\mu}{\pi}U'(z) = {} & \Big[\varphi_{zz}(r\varphi_r-\chi_z)\Big]_{r=0}^{r=a}+\Big[r\varphi_{rz}\Big(\frac{1}{r}\chi_r-\varphi_z\Big)\Big]_{r=0}^{r=a} \\
& +\int_0^a\Big\{\varphi_{zz}[\chi_{rz}-(r\varphi_r)_r]+\varphi_{rz}(r\varphi_{rz}-\chi_{zz}) \\
& +r\varphi_{rz}\Big[\varphi_{rz}-\Big(\frac{1}{r}\chi_r\Big)_r\Big]+(r\varphi_r)_r\Big(\frac{1}{r}\chi_{rz}-\varphi_{zz}\Big)\Big\}\,\mathrm{d}r.
\end{aligned}
\tag{5.6}
$$

Those terms in (5.6) which involve boundary values of derivatives of φ and χ may be simplified with the aid of the boundary conditions (4.29), (4.30) and (4.34). After such a simplification and a rearrangement of the integrand, (5.6) furnishes

$$
\begin{aligned}
-\frac{2\mu}{\pi}U'(z) = {} & -a^2\varphi_{zz}^2(a,z) \\
& +\int_0^a\Big[\chi_{rz}\nabla^2\varphi+2r\varphi_{rz}^2-\varphi_{rz}\Big(\chi_{rr}-\frac{1}{r}\chi_r+\chi_{zz}\Big)-2(r\varphi_r)_r\varphi_{zz}\Big]\mathrm{d}r.
\end{aligned}
\tag{5.7}
$$

The differential equations (4.27), (4.28), together with (5.7), then imply

$$
-\frac{2\mu}{\pi}U'(z) = -a^2\varphi_{zz}^2(a,z)+\int_0^a[(1-\sigma)r(\nabla^2\varphi)^2+2r\varphi_{rz}^2-2(r\varphi_r)_r\varphi_{zz}]\,\mathrm{d}r. \tag{5.8}
$$

Since $U(l) = 0$, integration of (5.8) gives

$$
\begin{aligned}
\frac{2\mu}{\pi}U(\zeta) = \int_\zeta^l\Big\{ & -a^2\varphi_{zz}^2(a,z) \\
& +\int_0^a\Big[(1-\sigma)r\varphi_{zz}^2+(1-\sigma)\frac{1}{r}(r\varphi_r)_r^2+2r\varphi_{rz}^2-2\sigma(r\varphi_r)_r\varphi_{zz}\Big]\mathrm{d}r\Big\}\,\mathrm{d}z,
\end{aligned}
\tag{5.9}
$$

for any ζ between 0 and l.

The expression (5.9) does not involve the stress function χ; apart from the boundary term $-a^2\varphi_{zz}^2(a,z)$, (5.9) is analogous to the corresponding formula in plane strain.†

6. AN AUXILIARY QUADRATIC FUNCTIONAL

For our ultimate purposes, it is more convenient to use a quadratic functional similar to, but simpler than, the strain energy $U(\zeta)$ discussed in the preceding section. Define $V(\zeta)$ by

$$
\frac{2\mu}{\pi}V(\zeta) = \int_\zeta^l\Big\{-a^2\varphi_{zz}^2(a,z)+(1-\sigma)\int_0^a\Big[r\varphi_{zz}^2+2r\varphi_{rz}^2+\frac{1}{r}(r\varphi_r)_r^2\Big]\mathrm{d}r\Big\}\mathrm{d}z, \tag{6.1}
$$

for $0 \le \zeta \le l$. Reference to (5.9) shows that when $\sigma = 0$, $U(\zeta) \equiv V(\zeta)$; we shall examine the relation between U and V for non-zero values of Poisson's ratio later in this section.

†For $\sigma = 0$, (5.9) may be compared with (3.1) of [3].

To simplify the notation and to make clear the analogy between the subsequent argument in the present paper and that given in [3] for plane strain, it is useful to introduce a suitable scalar product for functions continuous on the interval $0 \le r \le a$. For any two such functions f and g, define

$$(f, g) = -a^2 f(a) g(a) + (1-\sigma)\int_0^a r f(r) g(r)\,\mathrm{d}r. \tag{6.2}$$

In view of (4.36), the stress function φ satisfies the condition

$$(\varphi, 1) = 0, \qquad 0 \le z \le l. \tag{6.3†}$$

In general, the inequality $(f, f) \ge 0$ will *not* hold for an arbitrary continuous function f. We will make use of the following alternative sufficient conditions for the positivity of (f, f):

(a) If f is continuous on $[0, a]$ and $f(a) = 0$, then $(f, f) \ge 0$, with strict inequality holding unless $f \equiv 0$.

(b) If f is continuous on $[0, a]$ and $(f, 1) = 0$, then $(f, f) \ge 0$, with strict inequality holding unless $f \equiv 0$.

The result in (a) is immediate from (6.2) with $f = g$. The proof of (b) involves a straightforward application of the Schwarz inequality to the integral appearing in the definition of (f, f) and depends on the fact that $\sigma < \frac{1}{2}$.

In terms of the scalar product (6.2), (6.1) can be written in the form

$$\frac{2\mu}{\pi} V(\zeta) = \int_\zeta^l \left[(\varphi_{zz}, \varphi_{zz}) + 2(\varphi_{rz}, \varphi_{rz}) + (1-\sigma)\int_0^a \frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r \right] \mathrm{d}z. \tag{6.4}$$

Since (6.3) implies that

$$(\varphi_z, 1) = (\varphi_{zz}, 1) = 0, \qquad 0 \le z \le l, \tag{6.5}$$

and since φ_{zr} vanishes at $r = a$, it may be concluded that each of the three terms in the integrand of (6.4) is nonnegative.

In the following section we shall derive an exponential decay formula for $V(z)$. To carry out this derivation, and to show that such an exponential decay law can in turn be used to estimate $U(z)$, we require the following three properties of V.

(i) $$\frac{2\mu}{\pi} V'(z) = -(\varphi_{zz}, \varphi_{zz}) - 2(\varphi_{rz}, \varphi_{rz}) - (1-\sigma)\int_0^a \frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r, \tag{6.6}$$

(ii) $$\frac{2\mu}{\pi}\int_z^l V(\zeta)\,\mathrm{d}\zeta = (\varphi_z, \varphi_z) - (\varphi, \varphi_{zz}) + (\varphi_r, \varphi_r), \tag{6.7}$$

(iii) $$0 < \left(1 - \frac{|\sigma|}{1-\sigma}\right) V(z) \le U(z) \le \left(1 + \frac{|\sigma|}{1-\sigma}\right) V(z) \tag{6.8}$$

(6.6), (6.7) and (6.8) hold for $0 \le z \le l$.

Property (i) follows immediately from (6.4) upon differentiation with respect to ζ. To establish property (ii), we proceed from (6.6). Using primes to indicate differentiation

† To avoid excessively cumbersome formulas, we do not explicitly indicate that (f, g) depends on z when f and g are functions of z as well as r.

with respect to z, we have

$$\begin{aligned}(\varphi_{zz},\varphi_{zz}) &= (\varphi_z,\varphi_{zz})'-(\varphi_z,\varphi_{zzz})\\ &= \tfrac{1}{2}(\varphi_z,\varphi_z)''-(\varphi,\varphi_{zzz})'+(\varphi,\varphi_{zzzz})\\ &= \tfrac{1}{2}(\varphi_z,\varphi_z)''-(\varphi,\varphi_{zz})''+(\varphi_z,\varphi_{zz})'+(\varphi,\varphi_{zzzz})\\ &= (\varphi_z,\varphi_z)''-(\varphi,\varphi_{zz})''+(\varphi,\varphi_{zzzz})\end{aligned} \tag{6.9}$$

Next,

$$\begin{aligned}(\varphi_{rz},\varphi_{rz}) &= (\varphi_r,\varphi_{rz})'-(\varphi_r,\varphi_{rzz})\\ &= \tfrac{1}{2}(\varphi_r,\varphi_r)''-(1-\sigma)\int_0^a r\varphi_r\varphi_{rzz}\,\mathrm{d}r\\ &= \tfrac{1}{2}(\varphi_r,\varphi_r)''+(1-\sigma)\int_0^a \varphi(r\varphi_{rzz})_r\,\mathrm{d}r\end{aligned} \tag{6.10}$$

Substituting (6.9) and (6.10) into (6.6), we find

$$\begin{aligned}\frac{2\mu}{\pi}V'(z) = &-[(\varphi_z,\varphi_z)-(\varphi,\varphi_{zz})+(\varphi_r,\varphi_r)]''\\ &-(\varphi,\varphi_{zzzz})-2(1-\sigma)\int_0^a \varphi(r\varphi_{rzz})_r\,\mathrm{d}r-(1-\sigma)\int_0^a \frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r.\end{aligned} \tag{6.11}$$

Integration by parts and the differential equations and boundary conditions satisfied by φ, χ can be used to show that

$$2(1-\sigma)\int_0^a \varphi(r\varphi_{rzz})_r\,\mathrm{d}r+(1-\sigma)\int_0^a \frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r = -(\varphi,\varphi_{zzzz}),$$

so that (6.11) becomes

$$\frac{2\mu}{\pi}V'(z) = -[(\varphi_z,\varphi_z)-(\varphi,\varphi_{zz})+(\varphi_r,\varphi_r)]''. \tag{6.12}$$

Integrating twice and using $V(l) = \varphi(r,l) = \varphi_z(r,l) = \varphi_r(r,l) = 0$, we find

$$\frac{2\mu}{\pi}\int_\zeta^l V(z)\,\mathrm{d}z = (\varphi_z,\varphi_z)-(\varphi,\varphi_{zz})+(\varphi_r,\varphi_r), \tag{6.13}$$

which is precisely (6.7).

The inequalities (6.8) show that an upper bound for either of the two functionals $U(z)$, $V(z)$ supplies an upper bound for the other. To prove (6.8), we first observe that (5.8) can be written in the form

$$-\frac{2\mu}{\pi}U'(z) = (\varphi_{zz},\varphi_{zz})+\frac{2}{1-\sigma}(\varphi_{rz},\varphi_{rz})+(1-\sigma)\int_0^a \frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r-2\sigma\int_0^a (r\varphi_r)_r\varphi_{zz}\,\mathrm{d}r, \tag{6.14}$$

so that, with the aid of (6.6),

$$-\frac{2\mu}{\pi}U'(z)+\frac{2\mu}{\pi}V'(z) = \frac{2\sigma}{1-\sigma}(\varphi_{rz},\varphi_{rz})-2\sigma\int_0^a (r\varphi_r)_r\varphi_{zz}\,\mathrm{d}r.$$

Next, set

$$I = (1-\sigma)\int_0^a (r\varphi_r)_r \varphi_{zz}\,\mathrm{d}r. \tag{6.15}$$

Then

$$-\frac{|\sigma|}{1-\sigma}[2(\varphi_{rz},\varphi_{rz})+2|I|] \leq -\frac{2\mu}{\pi}U'(z)+\frac{2\mu}{\pi}V'(z) \leq \frac{|\sigma|}{1-\sigma}[2(\varphi_{rz},\varphi_{rz})+2|I|]. \tag{6.16}$$

It remains to estimate $|I|$. To this end, define ψ through

$$\varphi_{zz}(r,z) = \varphi_{zz}(a,z)+\psi(r,z). \tag{6.17}$$

Since $\varphi_r(a,z) \equiv 0$, (6.15) can be written as

$$I = (1-\sigma)\int_0^a (r\varphi_r)_r\psi\,\mathrm{d}r. \tag{6.18}$$

Because of the inequality $|xy| \leq \frac{1}{2}(x^2+y^2)$, (6.18) implies

$$|I| \leq (1-\sigma)\int_0^a r^{-\frac{1}{2}}|(r\varphi_r)_r|r^{\frac{1}{2}}|\psi|\,\mathrm{d}r \leq \frac{1-\sigma}{2}\int_0^a \frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r+\frac{1-\sigma}{2}\int_0^a r\psi^2\,\mathrm{d}r. \tag{6.19}$$

From (6.17) and the second of (6.5), it follows easily that

$$(1-\sigma)\int_0^a r\psi^2\,\mathrm{d}r = (\varphi_{zz},\varphi_{zz})-\frac{1+\sigma}{2}a^2\varphi_{zz}^2(a,z) \leq (\varphi_{zz},\varphi_{zz}),$$

so that (6.19) implies

$$|I| \leq \frac{1-\sigma}{2}\int_0^a \frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r+\tfrac{1}{2}(\varphi_{zz},\varphi_{zz}). \tag{6.20}$$

From (6.20), it follows with the aid of (6.6) that

$$2(\varphi_{rz},\varphi_{rz})+2|I| \leq 2(\varphi_{rz},\varphi_{rz})+(1-\sigma)\int_0^a \frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r+(\varphi_{zz},\varphi_{zz}) = -\frac{2\mu}{\pi}V'(z). \tag{6.21}$$

Substitution from (6.21) into (6.16) gives

$$-\frac{|\sigma|}{1-\sigma}\left[-\frac{2\mu}{\pi}V'(z)\right] \leq -\frac{2\mu}{\pi}U'(z)+\frac{2\mu}{\pi}V'(z) \leq \frac{|\sigma|}{1-\sigma}\left[-\frac{2\mu}{\pi}V'(z)\right],$$

from which

$$\left(1-\frac{|\sigma|}{1-\sigma}\right)[-V'(z)] \leq -U'(z) \leq \left(1+\frac{|\sigma|}{1-\sigma}\right)[-V'(z)].$$

Integration from z to l, together with $V(l) = U(l) = 0$, finally furnishes

$$\left(1-\frac{|\sigma|}{1-\sigma}\right)V(z) \leq U(z) \leq \left(1+\frac{|\sigma|}{1-\sigma}\right)V(z). \tag{6.22}$$

Since $\sigma < \frac{1}{2}$ implies $1-|\sigma|(1-\sigma)^{-1} > 0$, the proof of (6.8) is complete.

7. ENERGY DECAY ESTIMATES

Properties (i) and (ii) of $V(z)$, as expressed by (6.6) and (6.7), are strictly analogous to the corresponding formulas derived in [3] for the case of plane strain.† It is therefore natural to expect that an argument similar to that used in [3] can be applied here to derive an exponential decay inequality for $V(z)$. In the present section we shall show that

$$V(z) \leq 2V(0)\exp(-2kz), \tag{7.1}$$

where the decay constant k is characterized as a root of a certain transcendental equation involving Bessel functions. From the two inequalities in (6.8), it then follows immediately that the strain energy $U(z)$ satisfies

$$U(z) \leq 2\left(\frac{1-\sigma+|\sigma|}{1-\sigma-|\sigma|}\right)U(0)\exp(-2kz), \qquad 0 \leq z \leq l. \tag{7.2}$$

We turn now to the proof of (7.1), which we carry out in two steps.

First, let k be an arbitrary positive constant and define S by

$$S(z) = V(z)+2k\int_z^l V(\zeta)\,\mathrm{d}\zeta, \qquad 0 \leq z \leq l. \tag{7.3}$$

Then, using (7.3), (6.6) and (6.7)

$$\left.\begin{aligned} S'(z)+2kS(z) &= V'(z)+4k^2\int_z^l V(\zeta)\,\mathrm{d}\zeta \\ &= -\frac{\pi}{2\mu}\{(\varphi_{zz},\varphi_{zz})+2(\varphi_{rz},\varphi_{rz})+(1-\sigma)\int_0^a \frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r \\ &\quad -4k^2(\varphi_z,\varphi_z)+4k^2(\varphi,\varphi_{zz})-4k^2(\varphi_r,\varphi_r)\} \\ &= -\frac{\pi}{2\mu}\{(\varphi_{zz}+2k^2\varphi,\varphi_{zz}+2k^2\varphi)+(1-\sigma)\int_0^a \frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r \\ &\quad -4k^2(\varphi_r,\varphi_r)-4k^4(\varphi,\varphi) \\ &\quad +2(\varphi_{rz},\varphi_{rz})-4k^2(\varphi_z,\varphi_z)\}. \end{aligned}\right\} \tag{7.4}$$

Since φ and φ_{zz} respectively satisfy (6.3) and the second of (6.5), the linear combination $\varphi_{zz}+2k^2\varphi$ satisfies $(\varphi_{zz}+2k^2\varphi, 1) = 0$, and therefore

$$(\varphi_{zz}+2k^2\varphi, \varphi_{zz}+2k^2\varphi) \geq 0, \tag{7.5}$$

so that (7.4) implies

$$\begin{aligned} S'(z)+2kS(z) \leq -\frac{\pi}{2\mu}\Bigg\{&(1-\sigma)\int_0^a \frac{1}{r}(r\varphi_r)_r^2\,\mathrm{d}r-4k^2(\varphi_r,\varphi_r) \\ &-4k^4(\varphi,\varphi)+2(\varphi_{rz},\varphi_{rz})-4k^2(\varphi_z,\varphi_z)\Bigg\}. \end{aligned} \tag{7.6}$$

†With a suitable redefinition of the scalar product appearing in (6.7), this equation becomes identical with equation (3.9) of [3].

To determine a positive value of k that makes the right side of (7.6) non-positive, we make use of the following two lemmas concerning eigenvalue problems.

LEMMA I. *Let ψ be continuously differentiable on* $[0, a]$, *and suppose that*

$$\psi(a) = 0, \qquad \psi(r) = O(r^2), \qquad \psi_{,r}(r) = O(r), \quad \text{as } r \to 0.$$

Then

$$\int_0^a r^{-1}\psi_r^2 \, dr \geq \lambda_{\mathrm{I}} \int_0^a r^{-1}\psi^2 \, dr, \tag{7.7}$$

where λ_{I} is the smallest eigenvalue of the problem

$$(r^{-1}\tilde{\psi}_{,r})_{,r} + \lambda r^{-1}\tilde{\psi} = 0 \quad on\ 0 < r \leq a, \tag{7.8}$$

$$\tilde{\psi} = O(r) \quad as\ r \to 0, \qquad \tilde{\psi}(a) = 0. \tag{7.9}$$

LEMMA II. *Let θ be continuously differentiable on* $[0, a]$, *and suppose that* $(\theta, 1) = 0$. *Then*

$$\int_0^a r\theta_r^2 \, dr \geq \lambda_{\mathrm{II}}(\theta, \theta), \tag{7.10}$$

where λ_{II} is the smallest positive *eigenvalue of the problem*

$$(r\tilde{\theta}_{,r})_{,r} + (1-\sigma)\lambda r\tilde{\theta} = 0 \quad on\ 0 \leq r \leq a, \tag{7.11}$$

$$\tilde{\theta}_{,r}(a) + \lambda a\tilde{\theta}(a) = 0, \qquad \tilde{\theta}(r) = O(1) \quad as\ r \to 0. \tag{7.12}$$

These lemmas are analogous to results employed in [1], [3], [4].†

We apply (7.7) and (7.10) repeatedly in (7.6) as follows. In Lemma I, choose $\psi = r\varphi_{,r}$ to conclude that

$$(1-\sigma)\int_0^a \frac{1}{r}(r\varphi_{,r})_{,r}^2 \, dr \geq (1-\sigma)\lambda_{\mathrm{I}} \int_0^a \frac{1}{r}(r\varphi_{,r})^2 \, dr = \lambda_{\mathrm{I}}(\varphi_{,r}, \varphi_{,r}). \tag{7.13}$$

Since $(\varphi_{,z}, 1) = 0$, the choice $\theta = \varphi_{,z}$ in Lemma II is permissible, and it may be concluded that

$$(\varphi_{,zr}, \varphi_{,zr}) = (1-\sigma)\int_0^a r\varphi_{,zr}^2 \, dr \geq (1-\sigma)\lambda_{\mathrm{II}}(\varphi_{,z}, \varphi_{,z}). \tag{7.14}$$

Furthermore $(\varphi, 1) = 0$ so that Lemma II with $\theta = \varphi$ may be used to show that

$$(\varphi_{,r}, \varphi_{,r}) = (1-\sigma)\int_0^a r\varphi_{,r}^2 \, dr \geq (1-\sigma)\lambda_{\mathrm{II}}(\varphi, \varphi). \tag{7.15}$$

From (7.13), (7.14) and (7.6)

$$\begin{aligned} S'(z) + 2kS(z) \leq -\frac{\pi}{2\mu}\{&(\lambda_{\mathrm{I}} - 4k^2)(\varphi_{,r}, \varphi_{,r}) - 4k^4(\varphi, \varphi) \\ &+ [2(1-\sigma)\lambda_{\mathrm{II}} - 4k^2](\varphi_{,z}, \varphi_{,z})\}. \end{aligned} \tag{7.16}$$

† Lemmas I and II may be established by appropriate adaptations of Jacobi's method of multiplicative variation. See [12], pp. 458–459.

If k is now required to satisfy

$$\lambda_{\mathrm{I}} - 4k^2 > 0, \tag{7.17}$$

(7.15) may be used in (7.16) to obtain

$$S'(z) + 2kS(z) \leq -\frac{\pi}{2\mu}\{(\lambda_{\mathrm{I}} - 4k^2)(1-\sigma)\lambda_{\mathrm{II}} - 4k^4\}(\varphi, \varphi) + [2(1-\sigma)\lambda_{\mathrm{II}} - 4k^2](\varphi_z, \varphi_z)\}. \tag{7.18}$$

If k is further restricted so that

$$(\lambda_{\mathrm{I}} - 4k^2)(1-\sigma)\lambda_{\mathrm{II}} - 4k^4 \geq 0, \tag{7.19}$$

$$2(1-\sigma)\lambda_{\mathrm{II}} - 4k^2 \geq 0, \tag{7.20}$$

(7.18) becomes

$$S'(z) + 2kS(z) \leq 0,$$

from which

$$S(z) \leq S(0)\exp(-2kz). \tag{7.21}$$

The polynomial in k appearing in (7.19) is monotone decreasing in k for $k > 0$ and has one positive root k^* given by

$$k^* = \{\tfrac{1}{2}[\sqrt{\{(1-\sigma)^2\lambda_{\mathrm{II}}^2 + (1-\sigma)\lambda_{\mathrm{I}}\lambda_{\mathrm{II}}\}} - (1-\sigma)\lambda_{\mathrm{II}}]\}^{\frac{1}{2}}. \tag{7.22}$$

The largest value of k satisfying the three restrictions (7.17), (7.19) and (7.20) is given by

$$k = \min\left(\frac{\sqrt{\lambda_{\mathrm{I}}}}{2}, \sqrt{\left[\frac{(1-\sigma)\lambda_{\mathrm{II}}}{2}\right]}, k^*\right). \tag{7.23}$$

It is easy to show from (7.22) that $k^* \leq \frac{1}{2}\sqrt{\lambda_{\mathrm{I}}}$, so that (7.23) may be replaced by

$$k = \min\left(\sqrt{\left[\frac{(1-\sigma)\lambda_{\mathrm{II}}}{2}\right]}, k^*\right). \tag{7.24}$$

Direct calculation also shows that (7.24) may be written in the form

$$k = \begin{cases} \left[\dfrac{\sqrt{\{(1-\sigma)^2\lambda_{\mathrm{II}}^2 + (1-\sigma)\lambda_{\mathrm{I}}\lambda_{\mathrm{II}}\}} - (1-\sigma)\lambda_{\mathrm{II}}}{2}\right]^{\frac{1}{2}} & \text{if } \lambda_{\mathrm{I}} < 3(1-\sigma)\lambda_{\mathrm{II}} \\ \sqrt{\left[\dfrac{(1-\sigma)\lambda_{\mathrm{II}}}{2}\right]} & \text{if } \lambda_{\mathrm{I}} \geq 3(1-\sigma)\lambda_{\mathrm{II}} \end{cases} \tag{7.25}$$

Thus the exponential decay inequality (7.21), with k given by (7.25), has been established for $S(z)$.

The second step in the proof consists in showing that (7.21) implies (7.1). This portion of the argument is identical with the corresponding one employed in [3]. From (7.3), (7.21) it follows that

$$V(z) \leq S(z) \leq S(0)\exp(-2kz), \tag{7.26}$$

so that it is only necessary to estimate $S(0)$. Upon inserting (7.3) into (7.21), it is found that

$$-\frac{d}{dz}[e^{-2kz}\int_z^l V(\zeta)\,d\zeta] \le S(0)\,e^{-4kz}. \tag{7.27}$$

Integration from $z = 0$ to $z = l$ yields

$$\int_0^l V(\zeta)\,d\zeta \le \frac{S(0)}{4k}(1-e^{-4kl}) = \frac{1}{4k}\left[V(0)+2k\int_0^l V(z)\,dz\right](1-e^{-4kl}). \tag{7.28}$$

When this inequality is solved for the integral of V, it is found that

$$2k\int_0^l V(z)\,dz \le \frac{1-e^{-4kl}}{1+e^{-4kl}}V(0) \le V(0). \tag{7.29}$$

Together, (7.29) and (7.3) show that

$$S(0) \le 2V(0),$$

so (7.26) furnishes

$$V(z) \le 2V(0)\exp(-2kz).$$

This completes the derivation of (7.1).

8. STRESS ESTIMATES

At an interior point (r, θ, z) of the cylinder the stress components may be estimated in terms of the strain energy by methods based on mean value theorems of the theory of elasticity, [1], [2], [3], [4]. Let τ stand for any one of the four nonvanishing cylindrical components of the stress tensor $\boldsymbol{\tau}$. It is known† that

$$|\tau(r,z)| \le K_1\delta^{-\frac{3}{2}}\sqrt{[U(z-\delta)]}, \qquad 0 \le r < a, \quad 0 < z < l, \tag{8.1}$$

where δ is the distance from the point (r, θ, z) to the boundary of the cylinder, and K_1 is a constant depending on μ and σ.

Roseman [2] has shown that an interior estimate of the type (8.1) may be replaced by an estimate valid up to the boundary of the cylinder. His result, which is valid for a cylinder whose cross-section is bounded by a sufficiently smooth simple closed curve, makes it possible to assert that, in the present case,

$$|\tau(r,z)| \le K_2 a^{-\frac{3}{2}}\sqrt{[U(z-a/2)]}, \qquad 0 \le r \le a, \quad a \le z \le l, \tag{8.2}$$

where a is the radius of the cylinder and K_2 is a constant depending on the properties of the material.

Either of the inequalities (8.1) or (8.2), together with (7.2), shows that the stresses satisfy an inequality of the type

$$|\tau(r,z)| \le K\exp(-kz), \tag{8.3}$$

† See Lemma 2 and equation (4.15) of [4].

where K is a constant and k is given by (7.25). Thus k provides a lower bound for the rate of exponential decay of stresses away from the loaded end of the cylinder. Since our interest here is mainly directed toward this rate of decay, we do not discuss the explicit determination of the constants K_1 and K_2 appearing in (8.1) and (8.2), respectively†.

The total energy $U(0)$ appearing in (7.2) can be estimated by the application of an appropriate minimum principle, but we shall not carry out such a calculation here. Such estimates have been obtained in similar contexts in [3] and [4]. A different type of upper bound for the total strain energy for a cylinder of square cross-section has been constructed by Dou [13].

9. THE DECAY CONSTANT

The eigenvalue λ_{I} of the problem represented by (7.8), (7.9) is given by

$$\lambda_{\mathrm{I}} = s^2/a^2 \tag{9.1}$$

where s is the smallest positive root of

$$J_1(s) = 0, \tag{9.2}$$

and J_n denotes the Bessel function of order n. From the eigenvalue problem (7.11), (7.12), λ_{II} is determined as

$$\lambda_{\mathrm{II}} = t^2/a^2, \tag{9.3}$$

where t is the smallest positive root of

$$-\sqrt{(1-\sigma)}J_1[\sqrt{(1-\sigma)}t]+tJ_0[\sqrt{(1-\sigma)}t] = 0. \qquad (9.4)‡$$

Once λ_{I} and λ_{II} are determined, the decay rate k is computed in accordance with (7.25).

In [6], Horvay and Mirabal consider a semi-infinite circular cylinder under axisymmetric self-equilibrated end loads. Their analysis is based on a variational approximation, and it permits the approximate calculation of the rate of decay of stresses with what would appear to be substantial accuracy. In Table 1, numerical values of the decay rate obtained by these authors are compared with values of ak, with k computed from (7.25).

TABLE 1. DECAY CONSTANT k FOR VARIOUS VALUES OF POISSON'S RATIO

Poisson's ratio, σ	ak, with k computed from (7.25)	(decay constant) xa, from Horvay and Mirabal [6]
0	1·3	2·5*
$\frac{1}{4}$	1·4	2·7*§
$\frac{3}{10}$	1·4	2·7†
$\frac{1}{2}$	1·5	2·8‡

* Estimated from Fig. 2 of [6].
† Given to four decimal places in equation (8a) of [6].
‡ Given to three decimal places in equation (40b) of [6].
§ See also Table 9, p. 395, of [7].

† K_1 is given explicitly in [4].
‡ Roots of (9.4) were obtained with the aid of the tables in [14].

From the table it is observed that k is small by a factor of about one-half when compared with the results of Horvay and Mirabal, whose estimates in turn probably agree with the exact values to the first decimal place.

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Абстракт—Используются методы заключающие неравенства затухания знергии для определения краевой задачи круглого упругого цилиндра. Получаются в явном виде нижные пределы в выражениях козффициента Пуассона для скорости зкспотенциального затухания знергии. Они сравниваются с результатами других авторов.